Exponential martingales and the UI martingale property

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Theme: When is an exponential martingale an uniformly integrable martingale, and why is this important?

Agenda:

1. Outline of exponential martingales
2. Previous results on the martingale property
3. Applications to point processes
4. Open problems

Outline of exponential martingales

In the following, assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions.

Consider a local martingale M with $\Delta M > -1$ and initial value zero. The exponential martingale $\mathcal{E}(M)$ of M is the unique cadlag adapted solution in to the equation $X_t = 1 + \int_0^t X_{s-} \, \mathrm{d}M_s$, and is given by

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M^c]_t + \sum_{0<s\leq t} \log(1+\Delta M_s) - \Delta M_s \right).$$

$\mathcal{E}(M)$ is a nonnegative local martingale with initial value 1 and a supermartingale with $E\mathcal{E}(M)_t \leq 1$.

If $\mathcal{E}(M)$ is an UI martingale, $\mathcal{E}(M)_\infty$ is a nonnegative variable with unit mean, and we may define a probability measure $Q = \mathcal{E}(M)_\infty \cdot P$.

Girsanov's Theorem and its variants describe the martingales under the measure Q.

Recall that by the Doob-Meyer decomposition theorem, any increasing locally integrable process A has a compensator $\Pi_p^* A$ such that $A - \Pi_p^* A$ is a local martingale.

Given local martingales M and N, if $[N, M]$ is locally integrable, we define the predictable covariation as $\Pi_p^*[N, M]$. The Lenglart-Girsanov Theorem states that if $\langle N, M \rangle$ exists, then the process

$$N - \langle N, M \rangle$$

is a Q martingale, with $Q = \mathcal{E}(M)_\infty \cdot P$.

The conclusions from these observations are:

1. By changing the measure and applying the Lenglart-Girsanov Theorem, we can construct processes with certain martingale properties.
2. Given P and Q on the same probability space, if we can identify M such that $Q = \mathcal{E}(M)_\infty \cdot P$, we obtain an expression for the likelihood $\frac{\mathrm{d}Q}{\mathrm{d}P}$.
3. To succeed in these objectives, we need useful sufficient criteria to determine when $\mathcal{E}(M)$ is a UI martingale.

Previous results

When is an exponential martingale an UI martingale?

The most well-known sufficient criterion is (Novikov 1972): If M is a continuous local martingale and $\exp(\frac{1}{2}[M]_\infty)$ is integrable, then $\mathcal{E}(M)$ is an UI martingale.

A much stronger result is (Lepingle & Mémin 1978): If M is a local martingale with $\Delta M > -1$, define

$$B_t = \frac{1}{2}[M^c]_t + \sum_{0<s\leq t} (1+\Delta M_s)\log(1+\Delta M_s) - \Delta M_s.$$

$\mathcal{E}(M)$ is an UI martingale if $\exp(\Pi_p^* B_\infty)$ is integrable.

Because it holds that

$$(1+x)\log(1+x) - x \leq \tfrac{1}{2}x^2$$

whenever $x \geq 0$, the Lepingle-Mémin result implies Novikov's criterion for local martingales with nonnegative jumps, in particular for continuous local martingales. For $x > -1$, we only have

$$(1+x)\log(1+x) - x \leq x^2,$$

thus giving a weaker Novikov-type result in the general case.

Applications to point processes

Consider, on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, a positive predictable locally bounded process λ and a step process N with steps of unit size.

If $N_t - \int_0^t \lambda_s \, \mathrm{d}s$ is a local martingale, we say that N is a point process with intensity λ.

Assume that N is a standard Poisson process.

If there is M such that $\mathcal{E}(M)$ is an UI martingale and such that under $Q = \mathcal{E}(M)_\infty \cdot P$, N is a point process with intensity λ, then we have both constructed a point process with intensity λ, and “sort of” identified its likelihood $\mathcal{E}(M)$ with respect to a standard Poisson process.

In general, for point processes with different intensities, their distributions are singular. For example, for a Poisson process with constant intensity λ, $\frac{N_t}{t} \xrightarrow{\text{a.s.}} \lambda$ and so different Poisson processes are concentrated on disjoint sets.

Therefore, we cannot in general hope to find $\mathcal{E}(M)$ such that with $Q = \mathcal{E}(M)_\infty \cdot P$, Q and P are equivalent and N is a point process with given intensity under Q.

Instead, we will do the following. If we can find M such that $\mathcal{E}(M)$ is a martingale, corresponding to having $\mathcal{E}(M^t)$ an UI martingale, we can define $Q_t = \mathcal{E}(M)_t \cdot P$. We can then try to find M such that N is a point process with intensity λ on $[0, t]$ under Q_t.

Our plan for this is as follows:

1. Find candidate for M.
2. Obtain small but useful lemma for proving the martingale property of $\mathcal{E}(M)$.
3. Prove the martingale property for M for a suitable class of candidate intensities λ.

Define $M_t = N_t - t$, put $H = \lambda - 1$ and assume that $\mathcal{E}(H \cdot M)$ is an UI martingale. Define $Q = \mathcal{E}(H \cdot M)_\infty \cdot P$. Under P, M is a martingale. By the Lenglart-Girsanov Theorem, under Q, $M_t - \langle M, H \cdot M \rangle_t$ is a martingale. However,

$$\begin{aligned} M_t - \langle M, H \cdot M \rangle_t &= N_t - t - ((\lambda - 1) \cdot \langle M \rangle)_t \\ &= N_t - t - ((\lambda - 1) \cdot \Pi_p^*[M])_t \\ &= N_t - \int_0^t \lambda_s \, \mathrm{d}s. \end{aligned}$$

Therefore, under Q, N is a point process with intensity λ. Thus, our candidate local martingale is $(\lambda - 1) \cdot M$.

Lemma. Let M be a local martingale with $\Delta M > -1$, and let $\varepsilon > 0$. If $\mathcal{E}(M^{n\varepsilon} - M^{(n-1)\varepsilon})$ is an UI martingale for all n, then $\mathcal{E}(M)$ is a martingale.

Proof. By the supermartingale property, it suffices to show that $E\mathcal{E}(M)_{n\varepsilon} = 1$, $n \geq 1$. By elementary results on the quadratic covariation, the processes $M^{n\varepsilon} - M^{(n-1)\varepsilon}$ have pairwise zero quadratic covariation. Therefore,

$$\mathcal{E}(M)_{n\varepsilon} = \prod_{k=1}^{n} \mathcal{E}(M^{k\varepsilon} - M^{(k-1)\varepsilon}).$$

Using that $\mathcal{E}(M^{k\varepsilon} - M^{(k-1)\varepsilon})$ is $\mathcal{F}_{n\varepsilon}$ measurable for $k \leq n$, and $\mathcal{E}(M^{k\varepsilon} - M^{(k-1)\varepsilon})_{(k-1)\varepsilon} = 1$, we may show using our assumptions about the martingale properties of $\mathcal{E}(M^{n\varepsilon} - M^{(n-1)\varepsilon})$ that

$$E\prod_{k=1}^{n}\mathcal{E}(M^{k\varepsilon} - M^{(k-1)\varepsilon}) = E\prod_{k=1}^{n-1}\mathcal{E}(M^{k\varepsilon} - M^{(k-1)\varepsilon})$$

for all $n \geq 1$. Therefore, $E\mathcal{E}(M)_{n\varepsilon} = 1$. $\square$

Lemma. Let $L = H \cdot M$, $M_t = N_t - t$. Put

$$B_t = \frac{1}{2}[L^c]_t + \sum_{0<s\leq t} (1 + \Delta L_s) \log(1 + \Delta L_s) - \Delta L_s.$$

Then $\Pi_p^* B_t = \int_0^t (1 + H_s) \log(1 + H_s) - H_s \, \mathrm{d}s$.

Proof. Since M has paths of finite variation, $L^c = 0$. The result follows by recalling that $\Pi_p^* N_t = t$ and making the observation that $\Delta L_t = H_t \Delta N_t$. $\square$

Theorem. Assume that $\lambda_t \leq \alpha N_{t-} + \beta$ for some $\alpha, \beta > 0$. Then $\mathcal{E}((\lambda - 1) \cdot M)$ is a martingale.

Proof. Define $L^n = (H \cdot M)^{n\varepsilon} - (H \cdot M)^{(n-1)\varepsilon}$ and $H = \lambda - 1$. Then $L^n = H 1_{]\!](n-1)\varepsilon, n\varepsilon]\!]} \cdot M$. It suffices to prove that there is $\varepsilon > 0$ such that $\mathcal{E}(L^n)$ is an UI martingale for all n. By the Lepingle-Mémin result and the preceeding lemma, it suffices to prove

$$E \exp\left(\int_{(n-1)\varepsilon}^{n\varepsilon} \lambda_s \log \lambda_s \, \mathrm{d}s \right) < \infty.$$

Since $x \mapsto x \log x$ is nonpositive on $x \leq 1$ and increasing on $x \geq 1$, we find by $\lambda_t \leq \alpha N_{t-} + \beta$ and elementary inequalities that

$$
\begin{aligned}
& \int_{(n-1)\varepsilon}^{n\varepsilon} \lambda_s \log \lambda_s \,\mathrm{d}s \\
\leq \; & \int_{(n-1)\varepsilon}^{n\varepsilon} (\alpha N_{s-} + \beta) \log(\alpha N_{s-} + \beta) \,\mathrm{d}s \\
\leq \; & \varepsilon(\alpha N_t + \beta) \log(\alpha N_t + \beta) \\
\leq \; & 4\varepsilon\alpha N_t \log N_t .
\end{aligned}
$$

Thus, it suffices to prove

$$E \exp(4\varepsilon\alpha N_t \log N_t) < \infty$$

for some $\varepsilon > 0$. As N has a Poisson distribution, this holds if we pick $\varepsilon > 0$ small enough so that $4\varepsilon\alpha < 1$. $\square$

The result obtained:

When N is a standard Poisson process and λ is positive predictable with $\lambda_t \leq \alpha N_{t-} + \beta$, we can find a measure change Q_t such that under Q_t, N has intensity λ on $[0, t]$, and we have an explicit expression for the likelihood.

Observations:

1. This reveals that the Lepingle-Mémin criterion is strong: the result is not true when λ has greater than linear growth in N.
2. A benefit of working in the general theory is that λ may depend on other processes than N, for example diffusions. Such constructions are not always trivial when working on canonical spaces.

Open problems

1. The results yield existence of many point processes on $[0, t]$ through a measure change, but does not yield any point processes on $[0, \infty)$. Since distributiosn of point processes on $[0, \infty)$ are in general not equivalent, measure changes cannot be used to obtain the full existence. Is it possible to find a way to construct point processes on $[0, \infty)$ using the general theory instead of manipulations on canonical spaces?

2. The results obtained are sufficient (probably) to construct point processes on $[0, t]$ with an intensity which is the absolute value of an Ornstein-Uhlenbeck process. What diffusions, in general, can be used as intensities?

3. Consider a prospective intensity process which has the jump-diffusion specification

$$\mathrm{d}\lambda_t = \mu(t, \lambda_t)\,\mathrm{d}t + \sigma(t, \lambda_t)\,\mathrm{d}W_t - (\lambda_{t-} - c)\,\mathrm{d}N_t,$$

that is, the intensity is reset to a constant level c at every jump. Does there exist a point process process with such an intensity, and is the distribution equivalent to the standard Poisson process on $[0, t]$? This is not at all clear from current results.

Thank you