Exponential martingales: uniform integrability results and applications to point processes

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26 September, 2012
Agenda

1. Exponential martingales
2. Novikov-type criteria: Optimal constants
3. Novikov-type criteria: Some elementary proofs
4. Applications to point processes (joint with N. R. Hansen)
Exponential martingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions. Unless otherwise noted, all processes are adapted and have initial value zero. We recall some conventions and definitions.

- An FV process $A$ is integrable if $E(V_A)_{\infty}$ is finite.
- $A$ is locally integrable if $A^{T_n}$ is integrable for some sequence of stopping times $(T_n)$ increasing to infinity.
- Any locally integrable FV process $A$ has a compensator $\Pi^*_p A$: A predictable and locally integrable FV process such that $A - \Pi^*_p A$ is a local martingale.
Exponential martingales

- For a local martingale $M$, the quadratic variation $[M]$ is the unique increasing process such that $M^2 - [M]$ is a local martingale.
- $M$ is locally square integrable if and only if $[M]$ is locally integrable, and in the affirmative, we let $\langle M \rangle$ be the compensator of $[M]$.
- There exists a decomposition $M = M^c + M^d$, where $M^c$ is continuous and $M^d$ is purely discontinuous.
Exponential martingales

For a local martingale \( M \), \( \mathcal{E}(M) \) is the unique càdlàg solution to the SDE 
\[
Z_t = 1 + \int_0^t Z_s^- \, dM_s,
\]
and is given by
\[
\mathcal{E}(M)_t = \exp \left( M_t - \frac{1}{2} [M^c]_t \right) \prod_{0 < s \leq t} (1 + \Delta M_s) e^{-\Delta M_s}.
\]

If \( \Delta M > -1 \), we also have
\[
\mathcal{E}(M)_t = \exp \left( M_t - \frac{1}{2} [M^c]_t + \sum_{0 < s \leq t} \log(1 + \Delta M_s) - \Delta M_s \right).
\]
Exponential martingales

Some properties:

- $\mathcal{E}(M)$ is always a local martingale with initial value one.
- If $\Delta M \geq -1$, $\mathcal{E}(M)$ is a nonnegative supermartingale.
- If $\Delta M \geq -1$, $\mathcal{E}(M)$ is almost surely convergent.
- If $\Delta M \geq -1$, $\mathcal{E}(M)$ is an UI martingale if and only if $E\mathcal{E}(M)_\infty = 1$. 
Exponential martingales

Main problem. Finding sufficient criteria to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale.

Motivation:

- Likelihood inference for continuously observed stochastic processes.
- Explicit pricing measures in mathematical finance.
- Methods for existence of solutions to martingale problems / SDE’s.
- The problem is challenging and interesting in itself.
The most classical sufficient criterion:

**Theorem (Novikov, 1972).** Let $M$ be a continuous local martingale. If $E \exp(\frac{1}{2}[M]_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale. Also, the constant $\frac{1}{2}$ is optimal.
Results for local martingales with jumps:

**Theorem (Lepingle & Mémin, 1978).** Let $M$ be a local martingale with $\Delta M > -1$. Put $A_t = \frac{1}{2} [M^c]_t + \sum_{0 < s \leq t} (1 + \Delta M_s) \log(1 + \Delta M_s) - \Delta M_s$. If $A$ is locally integrable and $E \exp(\Pi^*_p A_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale.

**Theorem (Lepingle & Mémin, 1978).** Let $M$ be a local martingale with $\Delta M > -1$. Put $A_t = \frac{1}{2} [M^c]_t + \sum_{0 < s \leq t} \log(1 + \Delta M_s) - \Delta M_s/(1 + \Delta M_s)$. If $E \exp(A_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale.
Novikov-type criteria: Optimal constants

A simple observation *(Protter & Shimbo, 2008)*: As it holds that

\[(1 + x) \log(1 + x) - x \leq x^2 \text{ for } x > -1,\]

the previous theorems imply that if \(E \exp(\frac{1}{2}\langle M^c \rangle_\infty + \langle M^d \rangle_\infty)\) is finite, \(\mathcal{E}(M)\) is a uniformly integrable martingale.

This is a Novikov-type criterion for \(\mathcal{E}(M)\) to be a uniformly integrable martingale. **Questions:**

- Are the constants in front of \(\langle M^c \rangle\) and \(\langle M^d \rangle\) optimal?
- Why is there a 1 instead of a \(\frac{1}{2}\) in front of \(\langle M^d \rangle\)?
- Can similar results be obtained with \([M^d]\) instead of \(\langle M^d \rangle\)?
Novikov-type criteria: Optimal constants

For $a > -1$ with $a \neq 0$, define

$$\alpha(a) = \frac{(1 + a) \log(1 + a) - a}{a^2}$$

$$\beta(a) = \frac{(1 + a) \log(1 + a) - a}{(1 + a)a^2}$$

**Theorem 1.** Let $a \geq -1$ and assume $\Delta M_{1, (\Delta M \neq 0)} \geq a$. It holds that

$$E \exp\left(\frac{1}{2} \langle M^c \rangle_\infty + \alpha(a) \langle M^d \rangle_\infty\right) < \infty \Rightarrow E \mathcal{E}(M)_\infty = 1$$

$$E \exp\left(\frac{1}{2} [M^c]_\infty + \beta(a) [M^d]_\infty\right) < \infty \Rightarrow E \mathcal{E}(M)_\infty = 1,$$

where the former requires local square-integrability to make sense. All constants are optimal. Note that $\beta(-1) = \infty$, so no sufficient criterion exists in the case $a = -1$ for this case.
Novikov-type criteria: Optimal constants

Graph of the function $\alpha$: 

![Graph of the function $\alpha$](image-url)
Novikov-type criteria: Optimal constants

Graph of the function $\beta$: 

![Graph of the function $\beta$](image)
Novikov-type criteria: Optimal constants

Outline of proof. Sufficiency follows immediately from the results of Lepingle & Mémin once we observe that

\[ \alpha(a) = \inf\{ c \geq 0 \mid (1 + x) \log(1 + x) - x \leq cx^2 \text{ for } x \geq a \} \]

\[ \beta(a) = \inf\{ c \geq 0 \mid \log(1 + x) - x/(1 + x) \leq cx^2 \text{ for } x \geq a \} \]

Optimality is more involved. The proof considers \( a > 0, a = 0, -1 < a < 0 \) and \( a = -1 \) separately. We outline the strategy for optimality of \( \alpha(a) \) in the case \( a > 0 \).
Novikov-type criteria: Optimal constants

Let $a > 0$ and let $\varepsilon > 0$. We show that $E \exp((1 - \varepsilon)\alpha(a)\langle M \rangle_\infty) < \infty$ is insufficient to yield $E\mathcal{E}(M)_\infty = 1$.

Let $N$ be a standard Poisson process and let $b > 0$. Define

$$T_b = \inf\{t \geq 0 \mid N_t - (1 + b)t = -1\}$$

$$M_t = a(N_t^{T_b} - t \wedge T_b)$$

It holds that $N_{T_b} = (1 + b)T_b - 1$. By elementary calculations,

$$\mathcal{E}(M)_\infty = \frac{1}{1 + a} \exp(T_b((1 + b)\log(1 + a) - a))$$

$$\exp((\alpha(a) - \varepsilon)\langle M \rangle_\infty) = \exp(T_b a^2(1 - \varepsilon)\alpha(a))$$
Novikov-type criteria: Optimal constants

By optional stopping arguments, we obtain the desired counterexample by choosing $b \in (0, a)$ such that

$$(1 + b) \log(1 + a) - a \leq 1 - (1 + a)\frac{b}{a} + (1 + b) \log(1 + a) + (1 + b) \log \frac{b}{a}$$

$$a^2(1 - \varepsilon)\alpha(a) \leq (1 + b) \log(1 + b) - b,$$

and by elementary analysis, such a choice can be made.

Remaining cases:

- Optimality of $\alpha(a)$ for $-1 < a < 0$: more involved.
- Optimality of $\alpha(a)$ for $a = -1$ and $a = 0$: not difficult.
- Optimality of $\beta(a)$: Similar to optimality of $\alpha(a)$. 

□
Novikov-type criteria: Optimal constants

**Corollary 2.** Let $M$ be a local martingale with $\Delta M \geq 0$.

1. If $\exp(\frac{1}{2}[M]_\infty)$ is integrable, it holds that $\mathcal{E}(M)$ is a uniformly integrable martingale.

2. If $M$ is locally square integrable and $\exp(\frac{1}{2}\langle M \rangle_\infty)$ is integrable, it holds that $\mathcal{E}(M)$ is a uniformly integrable martingale.

Both the constants and the requirement on the jumps are optimal.
Novikov-type criteria: Some elementary proofs

Questions:

- Can Corollary 2 be proved using elementary methods?
- Can Corollary 2 be extended?
Novikov-type criteria: Some elementary proofs

Current results. Similarly to (Krylov, 2009):

Theorem 3. Assume $\Delta M \geq 0$. It holds that $E\mathcal{E}(M)_\infty = 1$ if only

$$\liminf_{\varepsilon \to 0} \varepsilon \log E \exp\left((1 - \varepsilon)\frac{1}{2}[M]_\infty\right) < \infty$$

Theorem 4. Assume $\Delta M \geq 0$ and let $M$ be quasi-left-continuous. It holds that $E\mathcal{E}(M)_\infty = 1$ if only

$$\liminf_{\varepsilon \to 0} \varepsilon \log E \exp\left((1 - \varepsilon)\frac{1}{2}\left(\alpha[M]_\infty + (1 - \alpha)\langle M\rangle_\infty\right)\right) < \infty$$

Open problem. Extension of Theorem 4 to the non-QLC case.
Novikov-type criteria: Some elementary proofs

Outline of proof of Theorem 3. Note that for \( x \geq 0 \), we have

\[
0 \leq \log \frac{1 + \lambda x}{(1 + x)\lambda} \leq \frac{\lambda(1 - \lambda)}{2} x^2 \quad \text{when} \quad 0 \leq \lambda \leq 1
\]

\[
0 \leq \log \frac{(1 + x)^a}{1 + ax} \leq \frac{a(a - 1)}{2} x^2 \quad \text{when} \quad a \geq 1
\]

Let \( a, r > 1 \) and let \( s \) be the dual exponent to \( r \). Using Hölder’s inequality and the optional stopping theorem with \( \mathcal{E}(arM) \), we find that for any stopping time \( T \),

\[
E \mathcal{E}(M)^a_T \leq \left( E \exp \left( \frac{ar(ar - 1)}{2(r - 1)} [M]_\infty \right) \right)^{1/s}.
\]
Novikov-type criteria: Some elementary proofs

As $\inf_{a,r>1} \frac{ar(ar-1)}{2(r-1)} = \frac{1}{2}$, we conclude

$$E \exp((1 + \varepsilon)\frac{1}{2}[M]_{\infty}) < \infty \text{ for some } \varepsilon > 0 \Rightarrow E\mathcal{E}(M)_{\infty} = 1.$$ 

Next, note that by our assumptions, $E \exp((1 - \varepsilon)\frac{1}{2}[M]_{\infty})$ is finite for $\varepsilon > 0$. Therefore, for $0 < \lambda < 1$, $E \exp((1 + \varepsilon\lambda)\frac{1}{2}[\lambda M]_{\infty})$ is finite for some suitable $\varepsilon\lambda > 0$, so $E\mathcal{E}(\lambda M)_{\infty} = 1$. By Hölder’s inequality,

$$1 \leq \left(E\mathcal{E}(M)_{\infty}\right)^{\lambda} e^{\gamma\lambda(1-\lambda)/2} + \left(E\mathcal{E}(M)_{\infty} 1_{F_{\gamma}}\right)^{\lambda} \left(E \exp \left(\frac{\lambda}{2}[M]_{\infty}\right)\right)^{1-\lambda},$$

where $F_{\gamma} = ([M]_{\infty} > \gamma)$. Taking the limes inferior as $\lambda$ tends to one and letting $\gamma$ tend to infinity, we obtain $E\mathcal{E}(M)_{\infty} = 1$. □
Novikov-type criteria: Some elementary proofs

Outline of proof of Theorem 4. As for Theorem 2, except that we need

$$0 \leq \log \frac{1 + \lambda x + (1 + \sqrt{1 - \alpha x})^\lambda - (1 + \lambda \sqrt{1 - \alpha x})}{(1 + x)\lambda} \leq \alpha \frac{\lambda(1 - \lambda)}{2} x^2,$$

and we use Hölder’s inequality for triples with the processes

$$arM \text{ and } arM + W^{ar} - \Pi_p^* W^{ar} \text{ instead of } arM,$$

where $W^\beta = \sum_{0 < s \leq t} (1 + \Delta M_s)^\beta - (1 + \beta \Delta M_s)$, and secondly use

$$\lambda M + W^\lambda(\alpha) - \Pi_p^* W^\lambda(\alpha) \text{ instead of } \lambda M,$$

where $W^\lambda(\alpha) = \sum_{0 < s \leq t} (1 + \sqrt{1 - \alpha \Delta M_s})^\lambda - (1 + \lambda \sqrt{1 - \alpha \Delta M_s})$. \qed
Applications to point processes

**Definition.** We say that a $d$-dimensional nonexplosive point process $N$ has intensity $\lambda$ if $N_t^i - \int_0^t \lambda_s^i \, ds$ is a local martingale, $i \leq d$.

A statistical model for a counting process with intensity consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.
- A nonexplosive point process $N$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.
- A parametrized family $(\mu_\theta)_{\theta \in \Theta}$ of intensities.
- A corresponding family of probability measures $P_\theta$ such that under $P_\theta$, $N$ is a nonexplosive counting process with intensity $\mu_\theta$.

**Problem.** Given a family $(\mu_\theta)_{\theta \in \Theta}$, does there exist a statistical model corresponding to this family of candidate intensities? This is not a vacuous question, as many candidate intensities yield explosion.
Applications to point processes

Solution approach on canonical spaces. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be the space of nonexplosive point process trajectories endowed with the canonical \(\sigma\)-algebra and filtration, let \(N : \Omega \to \Omega\) be the identity and let \(P\) be such that \(N\) is a homogeneous Poisson process.

(Jacobsen, 2005) gives sufficient criteria on \(\mu_\theta\) to ensure that there exists a probability measure \(P_\theta\) equivalent to \(P\) such that under \(P_\theta\), \(N\) has intensity \(\mu_\theta\).

This yields the existence of nonexplosive point processes with intensity \(\mu_\theta\) and yields the existence of the statistical model.
Applications to point processes

Benefits of the canonical setting:

- Precise expressions for the likelihood in terms of the waiting time distributions of the point process with intensity $\mu_\theta$.
- Coupling arguments may be used to analyze non-explosion.

Drawbacks of the canonical setting:

- Only intensities depending on $N$ are covered.
- Arguments are often based on very technical manipulations of the canonical space and various conditional distributions, instead of for example modern martingale theory.

Alternative approach. Consider a general filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ and formulate all issues in terms of martingales.
Applications to point processes

A general problem statement. Assume given:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.
- A positive, predictable and locally bounded intensity process $\lambda$.
- A point process $N$ with intensity $\lambda$.
- A parametrized family $(\mu_\theta)_{\theta \in \Theta}$ of intensities.

We seek: Sufficient criteria on $\mu_\theta$ to ensure the existence of a probability measure $P_\theta$ equivalent to $P$ such that under $P_\theta$, $N$ has intensity $\mu_\theta$.

As corollaries, we obtain: Explicit expressions for the likelihood, criteria for existence of point processes with various intensities (corresponding to criteria for nonexplosion).
Applications to point processes

From now on, assume given:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions.
- Positive, predictable and locally bounded $d$-dimensional $\lambda, \mu$.
- A $d$-dimensional point process $N$ with intensity $\lambda$.

We define:

- $M^i_t = N^i_t - \int_0^t \lambda^i_s \, ds$.
- $\gamma^i_t = \mu^i_t (\lambda^i)_{t}^{-1}$.
- $H^i_t = \gamma^i_t - 1$.
- $H \cdot M = \sum_{i=1}^{d} \int_0^t H^i_s \, dM^i_s$. 
Applications to point processes

**Lemma 5.** Assume that $\mathcal{E}(H \cdot M)$ is a martingale. Let $t \geq 0$. With $Q_t$ being the measure with Radon-Nikodym derivative $\mathcal{E}(H \cdot M)_t$ with respect to $P$, $N$ is a counting process under $Q_t$ with intensity $1_{[0,t]} \mu + 1_{(t,\infty)} \lambda$.

**Conclusion.** In order to obtain the existence of the desired equivalent probability measures, we need criteria for the martingale property of $\mathcal{E}(H \cdot M)$. 
Applications to point processes

**Theorem 6.** Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t - u| \leq \varepsilon$, one of the following two conditions are satisfied:

$$E \exp \left( \sum_{i=1}^{d} \int_{u}^{t} (\gamma_s^i \log \gamma_s^i - (\gamma_s^i - 1)) \lambda_s^i \, ds \right) < \infty \quad \text{or}$$

$$E \exp \left( \sum_{i=1}^{d} \left( \int_{u}^{t} \lambda_s^i \, ds + \int_{u}^{t} \log_+ \gamma_s^i \, dN_s^i \right) \right) < \infty,$$

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.
Applications to point processes

**Corollary 7.** Let $\lambda = 1$. Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t - u| \leq \varepsilon$, one of the following two conditions are satisfied:

\[ E \exp \left( \sum_{i=1}^{d} \int_{u}^{t} \mu_s^i \log_+ \mu_s^i \, ds \right) < \infty \quad \text{or} \]

\[ E \exp \left( \sum_{i=1}^{d} \int_{u}^{t} \log_+ \mu_s^i \, dN_s^i \right) < \infty, \]

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.
Applications to point processes

Outline of proof of Theorem 6:

1. Argue that it suffices to show that $\mathcal{E}((H \cdot M)^t - (H \cdot M)^u)$ is a martingale for $|t - u| \leq \varepsilon$.

2. Decompose $\mu$ into large and small parts and show a related decomposition for exponential martingales.

3. Apply two theorems of (Lépingle & Mémin, 1978) to obtain the result.
Applications to point processes

**Example.** Let $\mu^i_t \leq \alpha + \beta \sum_{j=1}^{d} N^j_{t-}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

The Example shows that we may recover the classical affine criteria for non-explosion from the canonical case in the case of a general filtered space. This also extends the criterion from (Gjessing et al., 2010) from an “$L^p$”-criterion, $p > 1$, to an “$L^p$”-criterion, $p \geq 1$.

**Outline of proof.** To use the first moment condition, use that $E \exp(\varepsilon X \log X)$ is finite when $X$ is Poisson distributed and $0 < \varepsilon < 1$, choose $\varepsilon > 0$ such that $4\beta \varepsilon d < 1$. To use the second moment condition, use a Markov argument and that Poisson distributions have moments of all orders, choose $\varepsilon > 0$ such that $\beta \varepsilon d < 1$. 
Applications to point processes

**Example.** Consider $A : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{R}^d$, $B : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d, d)$ and $\sigma : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d, d)$. Assume that $A(\eta, \cdot)$, $B(\eta, \cdot)$ and $\sigma(\eta, \cdot)$ are continuous and bounded for $\eta \in \mathbb{N}_0^d$. Assume that $\sigma$ is positive definite. Assume that for $\eta \in \mathbb{N}_0^d$, there is $\delta, c > 0$ such that

\[
\sup_{t \geq 0} \|A(\eta, t)\|_2 \leq c\|\eta\|_1^{1-\delta}
\]

\[
\sup_{t \geq 0} \|\sigma(\eta, t)\|_2 \leq c\|\eta\|_1^{(1-\delta)/2}
\]

\[
\sup_{t \geq 0} \|B(\eta, t)\|_2 \leq c.
\]
Applications to point processes

Example, continued. Let $X$ be a solution to

$$dX_t = (A(N_t, Z_t) + B(N_t, Z_t)X_t) \, dt + \sigma(N_t, Z_t) \, dW_t,$$

where $W$ is a $d$-dimensional ($\mathcal{F}_t$) Brownian motion and $Z_t^i = t - T_{N_t}^i$, where $T_{N_t}^i$ is the $n$'th event time of $N^i$. Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be Lipschitz and put $\mu_t = \phi(X_t)$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

The example shows that we can use our results to construct counting processes where the intensity is driven by a SDE whose coefficients vary according to the history of the counting process.
Outline of proof. Note that conditionally on \( N \), the intensity has the distribution of a Gaussian process. Apply bounds for \( E \exp(c \|Z\|_{2}^{1+\varepsilon}) \), with \( Z \) \( d \)-dimensionally Gaussian and \( 0 < \varepsilon < 1 \), to obtain a bound for the conditional expectation

\[
E \left( \exp \left( t \sum_{i=1}^{d} \mu_{s}^{i} \log \mu_{s}^{i} \right) \middle| N \right).
\]

Use this to obtain a bound of the unconditional expectation varying continuously in \( s \), \( 0 \leq s \leq t \). Apply Jensen’s inequality and further estimates to obtain the result.
Applications to point processes

Example. Let $\phi_i : \mathbb{R} \to [0, \infty)$ for $i \leq d$ and $h_{ij} : \mathbb{R}_+ \to \mathbb{R}$ for $i, j \leq d$. Define

$$
\mu_t^i = \phi_i \left( \sum_{j=1}^{d} \int_0^{t^-} h_{ij}(t - s) \, dN_s^j \right).
$$

Assume that $\phi^i$ is Borel measurable, that $\phi_i(x) \leq |x|$ and that $h_{ij}$ is bounded. Then $\mathcal{E}(H \cdot M)$ is a martingale.

This is an example of a sufficient criterion for non-explosion for multidimensional Hawkes processes.
References


References


Thank you!