Exponential martingales: uniform integrability results and applications to point processes

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Agenda

1. Exponential martingales
2. Novikov-type criteria: Optimal constants
3. Novikov-type criteria: Some elementary proofs
4. Applications to point processes (joint with N. R. Hansen)

Exponential martingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions. Unless otherwise noted, all processes are adapted and have initial value zero. We recall some conventions and definitions.

- An FV process A is integrable if $E(V_A)_\infty$ is finite.
- A is locally integrable if A^{T_n} is integrable for some sequence of stopping times (T_n) increasing to infinity.
- Any locally integrable FV process A has a compensator $\Pi_p^* A$: A predictable and locally integrable FV process such that $A - \Pi_p^* A$ is a local martingale.

Exponential martingales

- For a local martingale M, the quadratic variation $[M]$ is the unique increasing process such that $M^2 - [M]$ is a local martingale.
- M is locally square integrable if and only if $[M]$ is locally integrable, and in the affirmative, we let $\langle M \rangle$ be the compensator of $[M]$.
- There exists a decomposition $M = M^c + M^d$, where M^c is continuous and M^d is purely discontinuous.

Exponential martingales

For a local martingale M, $\mathcal{E}(M)$ is the unique càdlàg solution to the SDE $Z_t = 1 + \int_0^t Z_{s-}\,\mathrm{d}M_s$, and is given by

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M^c]_t\right) \prod_{0<s\leq t} (1 + \Delta M_s) e^{-\Delta M_s}.$$

If $\Delta M > -1$, we also have

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M^c]_t + \sum_{0<s\leq t} \log(1 + \Delta M_s) - \Delta M_s\right).$$

Exponential martingales

Some properties:

- $\mathcal{E}(M)$ is always a local martingale with initial value one.
- If $\Delta M \geq -1$, $\mathcal{E}(M)$ is a nonnegative supermartingale.
- If $\Delta M \geq -1$, $\mathcal{E}(M)$ is almost surely convergent.
- If $\Delta M \geq -1$, $\mathcal{E}(M)$ is an UI martingale if and only if $E\mathcal{E}(M)_\infty = 1$.

Exponential martingales

Main problem. Finding sufficient criteria to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale.

Motivation:

- Likelihood inference for continuously observed stochastic processes.
- Explicit pricing measures in mathematical finance.
- Methods for existence of solutions to martingale problems / SDE's.
- The problem is challenging and interesting in itself.

Novikov-type criteria: Optimal constants

The most classical sufficient criterion:

Theorem (Novikov, 1972). Let M be a continuous local martingale. If $E\exp(\frac{1}{2}[M]_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale. Also, the constant $\frac{1}{2}$ is optimal.

Novikov-type criteria: Optimal constants

Results for local martingales with jumps:

Theorem (Lepingle & Mémin, 1978). Let M be a local martingale with $\Delta M > -1$. Put $A_t = \frac{1}{2}[M^c]_t + \sum_{0<s\leq t}(1+\Delta M_s)\log(1+\Delta M_s) - \Delta M_s$. If A is locally integrable and $E\exp(\Pi_p^* A_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale.

Theorem (Lepingle & Mémin, 1978). Let M be a local martingale with $\Delta M > -1$. Put $A_t = \frac{1}{2}[M^c]_t + \sum_{0<s\leq t}\log(1+\Delta M_s) - \Delta M_s/(1+\Delta M_s)$. If $E\exp(A_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale.

Novikov-type criteria: Optimal constants

A simple observation **(Protter & Shimbo, 2008)**: As it holds that

$$(1+x)\log(1+x) - x \leq x^2 \text{ for } x > -1,$$

the previous theorems imply that if $E\exp(\frac{1}{2}\langle M^c\rangle_\infty + \langle M^d\rangle_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale.

This is a Novikov-type criterion for $\mathcal{E}(M)$ to be a uniformly integrable martingale. **Questions:**

- Are the constants in front of $\langle M^c\rangle$ and $\langle M^d\rangle$ optimal?
- Why is there a 1 instead of a $\frac{1}{2}$ in front of $\langle M^d\rangle$?
- Can similar results be obtained with $[M^d]$ instead of $\langle M^d\rangle$?

Novikov-type criteria: Optimal constants

For $a > -1$ with $a \neq 0$, define

$$\alpha(a) = \frac{(1+a)\log(1+a) - a}{a^2}$$
$$\beta(a) = \frac{(1+a)\log(1+a) - a}{(1+a)a^2}$$

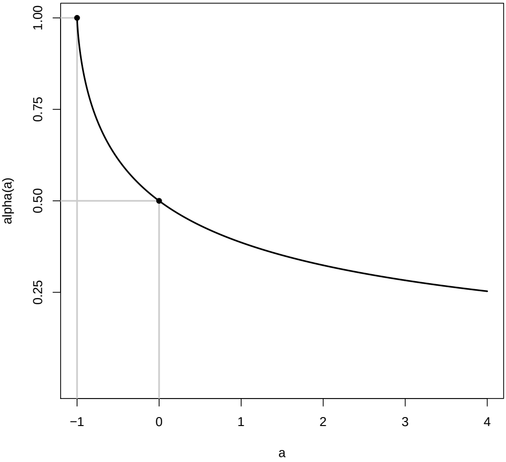
Theorem 1. Let $a \geq -1$ and assume $\Delta M 1_{(\Delta M \neq 0)} \geq a$. It holds that

$$E\exp(\tfrac{1}{2}\langle M^c\rangle_\infty + \alpha(a)\langle M^d\rangle_\infty) < \infty \Rightarrow E\mathcal{E}(M)_\infty = 1$$
$$E\exp(\tfrac{1}{2}[M^c]_\infty + \beta(a)[M^d]_\infty) < \infty \Rightarrow E\mathcal{E}(M)_\infty = 1,$$

where the former requires local square-integrability to make sense. All constants are optimal. Note that $\beta(-1) = \infty$, so no sufficient criterion exists in the case $a = -1$ for this case.

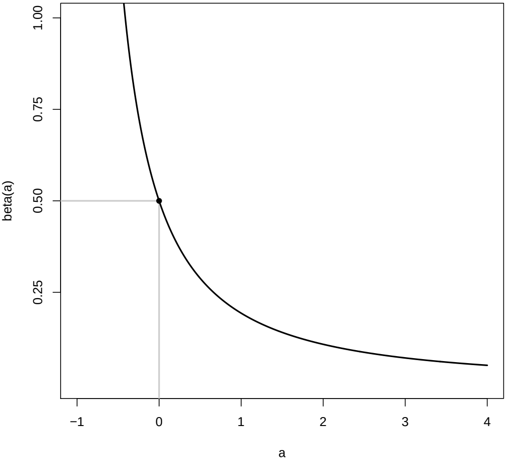
Novikov-type criteria: Optimal constants

Graph of the function α:



Novikov-type criteria: Optimal constants

Graph of the function β:



Novikov-type criteria: Optimal constants

Outline of proof. Sufficiency follows immediately from the results of Lepingle & Mémin once we observe that

$$\alpha(a) = \inf\{c \geq 0 \mid (1+x)\log(1+x) - x \leq cx^2 \text{ for } x \geq a\}$$
$$\beta(a) = \inf\{c \geq 0 \mid \log(1+x) - x/(1+x) \leq cx^2 \text{ for } x \geq a\}$$

Optimality is more involved. The proof considers $a > 0$, $a = 0$, $-1 < a < 0$ and $a = -1$ separately. We outline the strategy for optimality of $\alpha(a)$ in the case $a > 0$.

Novikov-type criteria: Optimal constants

Let $a > 0$ and let $\varepsilon > 0$. We show that $E\exp((1-\varepsilon)\alpha(a)\langle M\rangle_\infty) < \infty$ is insufficient to yield $E\mathcal{E}(M)_\infty = 1$.

Let N be a standard Poisson process and let $b > 0$. Define

$$T_b = \inf\{t \geq 0 \mid N_t - (1+b)t = -1\}$$
$$M_t = a(N_t^{T_b} - t \wedge T_b)$$

It holds that $N_{T_b} = (1+b)T_b - 1$. By elementary calculations,

$$\mathcal{E}(M)_\infty = \frac{1}{1+a}\exp(T_b((1+b)\log(1+a) - a))$$
$$\exp((\alpha(a)-\varepsilon)\langle M\rangle_\infty) = \exp(T_b a^2(1-\varepsilon)\alpha(a))$$

Novikov-type criteria: Optimal constants

By optional stopping arguments, we obtain the desired counterexample by choosing $b \in (0, a)$ such that

$$(1+b)\log(1+a) - a \leq 1 - (1+a)\tfrac{b}{a} + (1+b)\log(1+a) + (1+b)\log\tfrac{b}{a}$$
$$a^2(1-\varepsilon)\alpha(a) \leq (1+b)\log(1+b) - b,$$

and by elementary analysis, such a choice can be made.

Remaining cases:

- Optimality of $\alpha(a)$ for $-1 < a < 0$: more involved.
- Optimality of $\alpha(a)$ for $a = -1$ and $a = 0$: not difficult.
- Optimality of $\beta(a)$: Similar to optimality of $\alpha(a)$.

$\square$

Novikov-type criteria: Optimal constants

Corollary 2. Let M be a local martingale with $\Delta M \geq 0$.

1. If $\exp(\frac{1}{2}[M]_\infty)$ is integrable, it holds that $\mathcal{E}(M)$ is a uniformly integrable martingale.
2. If M is locally square integrable and $\exp(\frac{1}{2}\langle M\rangle_\infty)$ is integrable, it holds that $\mathcal{E}(M)$ is a uniformly integrable martingale.

Both the constants and the requirement on the jumps are optimal.

Novikov-type criteria: Some elementary proofs

Questions:

- Can Corollary 2 be proved using elementary methods?
- Can Corollary 2 be extended?

Novikov-type criteria: Some elementary proofs

Current results. Similarly to **(Krylov, 2009)**:

Theorem 3. Assume $\Delta M \geq 0$. It holds that $E\mathcal{E}(M)_\infty = 1$ if only

$$\liminf_{\varepsilon \to 0} \varepsilon \log E \exp((1-\varepsilon)\tfrac{1}{2}[M]_\infty) < \infty$$

Theorem 4. Assume $\Delta M \geq 0$ and let M be quasi-left-continuous. It holds that $E\mathcal{E}(M)_\infty = 1$ if only

$$\liminf_{\varepsilon \to 0} \varepsilon \log E \exp((1-\varepsilon)\tfrac{1}{2}(\alpha[M]_\infty + (1-\alpha)\langle M\rangle_\infty)) < \infty$$

Open problem. Extension of Theorem 4 to the non-QLC case.

Novikov-type criteria: Some elementary proofs

Outline of proof of Theorem 3. Note that for $x \geq 0$, we have

$$0 \leq \log \frac{1+\lambda x}{(1+x)^\lambda} \leq \frac{\lambda(1-\lambda)}{2} x^2 \text{ when } 0 \leq \lambda \leq 1$$
$$0 \leq \log \frac{(1+x)^a}{1+ax} \leq \frac{a(a-1)}{2} x^2 \text{ when } a \geq 1$$

Let $a, r > 1$ and let s be the dual exponent to r. Using Hölder's inequality and the optional stopping theorem with $\mathcal{E}(arM)$, we find that for any stopping time T,

$$E\mathcal{E}(M)_T^a \leq \left(E \exp\left(\frac{ar(ar-1)}{2(r-1)} [M]_\infty \right) \right)^{1/s}.$$

Novikov-type criteria: Some elementary proofs

As $\inf_{a,r>1} \frac{ar(ar-1)}{2(r-1)} = \frac{1}{2}$, we conclude

$$E\exp((1+\varepsilon)\tfrac{1}{2}[M]_\infty) < \infty \text{ for some } \varepsilon > 0 \Rightarrow E\mathcal{E}(M)_\infty = 1.$$

Next, note that by our assumptions, $E\exp((1-\varepsilon)\frac{1}{2}[M]_\infty)$ is finite for $\varepsilon > 0$. Therefore, for $0 < \lambda < 1$, $E\exp((1+\varepsilon_\lambda)\frac{1}{2}[\lambda M]_\infty)$ is finite for some suitable $\varepsilon_\lambda > 0$, so $E\mathcal{E}(\lambda M)_\infty = 1$. By Hölder's inequality,

$$1 \leq (E\mathcal{E}(M)_\infty)^\lambda e^{\gamma\lambda(1-\lambda)/2} + (E\mathcal{E}(M)_\infty 1_{F_\gamma})^\lambda \left(E\exp\left(\frac{\lambda}{2}[M]_\infty\right)\right)^{1-\lambda},$$

where $F_\gamma = ([M]_\infty > \gamma)$. Taking the limes inferior as λ tends to one and letting γ tend to infinity, we obtain $E\mathcal{E}(M)_\infty = 1$. □

Novikov-type criteria: Some elementary proofs

Outline of proof of Theorem 4. As for Theorem 2, except that we need

$$0 \leq \log \frac{1 + \lambda x + (1 + \sqrt{1-\alpha}x)^{\lambda} - (1 + \lambda\sqrt{1-\alpha}x)}{(1+x)^{\lambda}} \leq \alpha \frac{\lambda(1-\lambda)}{2} x^2,$$

and we use Hölder's inequality for triples with the processes

$$arM \text{ and } arM + W^{ar} - \Pi_p^* W^{ar} \text{ instead of } arM,$$

where $W^{\beta} = \sum_{0<s\leq t} (1 + \Delta M_s)^{\beta} - (1 + \beta \Delta M_s)$, and secondly use

$$\lambda M + W^{\lambda}(\alpha) - \Pi_p^* W^{\lambda}(\alpha) \text{ instead of } \lambda M,$$

where $W^{\lambda}(\alpha) = \sum_{0<s\leq t} (1 + \sqrt{1-\alpha}\Delta M_s)^{\lambda} - (1 + \lambda\sqrt{1-\alpha}\Delta M_s)$. □

Applications to point processes

Definition. We say that a d-dimensional nonexplosive point process N has intensity λ if $N_t^i - \int_0^t \lambda_s^i \,\mathrm{d}s$ is a local martingale, $i \leq d$.

A statistical model for a counting process with intensity consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A nonexplosive point process N on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A parametrized family $(\mu_\theta)_{\theta\in\Theta}$ of intensities.
- A corresponding family of probability measures P_θ such that under P_θ, N is a nonexplosive counting process with intensity μ_θ.

Problem. Given a family $(\mu_\theta)_{\theta\in\Theta}$, does there exist a statistical model corresponding to this family of candidate intensities? This is not a vacuous question, as many candidate intensities yield explosion.

Applications to point processes

Solution approach on canonical spaces. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be the space of nonexplosive point process trajectories endowed with the canonical σ-algebra and filtration, let $N : \Omega \to \Omega$ be the identity and let P be such that N is a homogeneous Poisson process.

(Jacobsen, 2005) gives sufficient criteria on μ_θ to ensure that there exists a probability measure P_θ equivalent to P such that under P_θ, N has intensity μ_θ.

This yields the existence of nonexplosive point processes with intensity μ_θ and yields the existence of the statistical model.

Applications to point processes

Benefits of the canonical setting:

- Precise expressions for the likelihood in terms of the waiting time distributions of the point process with intensity μ_θ.
- Coupling arguments may be used to analyze non-explosion.

Drawbacks of the canonical setting:

- Only intensities depending on N are covered.
- Arguments are often based on very technical manipulations of the canonical space and various conditional distributions, instead of for example modern martingale theory.

Alternative approach. Consider a general filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and formulate all issues in terms of martingales.

Applications to point processes

A general problem statement. Assume given:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A positive, predictable and locally bounded intensity process λ.
- A point process N with intensity λ.
- A parametrized family $(\mu_\theta)_{\theta\in\Theta}$ of intensities.

We seek: Sufficient criteria on μ_θ to ensure the existence of a probability measure P_θ equivalent to P such that under P_θ, N has intensity μ_θ.

As corollaries, we obtain: Explicit expressions for the likelihood, criteria for existence of point processes with various intensities (corresponding to criteria for nonexplosion).

Applications to point processes

From now on, assume given:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions.
- Positive, predictable and locally bounded d-dimensional λ, μ.
- A d-dimensional point process N with intensity λ.

We define:

- $M_t^i = N_t^i - \int_0^t \lambda_s^i \, \mathrm{d}s$.
- $\gamma_t^i = \mu_t^i (\lambda^i)_t^{-1}$.
- $H_t^i = \gamma_t^i - 1$.
- $H \cdot M = \sum_{i=1}^d \int_0^t H_s^i \, \mathrm{d}M_s^i$.

Applications to point processes

Lemma 5. Assume that $\mathcal{E}(H \cdot M)$ is a martingale. Let $t \geq 0$. With Q_t being the measure with Radon-Nikodym derivative $\mathcal{E}(H \cdot M)_t$ with respect to P, N is a counting process under Q_t with intensity $1_{[0,t]}\mu + 1_{(t,\infty)}\lambda$.

Conclusion. In order to obtain the existence of the desired equivalent probability measures, we need criteria for the martingale property of $\mathcal{E}(H \cdot M)$.

Applications to point processes

Theorem 6. Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t-u| \leq \varepsilon$, one of the following two conditions are satisfied:

$$E \exp\left(\sum_{i=1}^{d} \int_u^t (\gamma_s^i \log \gamma_s^i - (\gamma_s^i - 1))\lambda_s^i \, \mathrm{d}s\right) < \infty \quad \text{or}$$

$$E \exp\left(\sum_{i=1}^{d} \int_u^t \lambda_s^i \, \mathrm{d}s + \int_u^t \log_+ \gamma_s^i \, \mathrm{d}N_s^i\right) < \infty,$$

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Applications to point processes

Corollary 7. Let $\lambda = 1$. Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t - u| \leq \varepsilon$, one of the following two conditions are satisfied:

$$E \exp\left(\sum_{i=1}^{d} \int_u^t \mu_s^i \log_+ \mu_s^i \,\mathrm{d}s\right) < \infty \quad \text{or}$$

$$E \exp\left(\sum_{i=1}^{d} \int_u^t \log_+ \mu_s^i \,\mathrm{d}N_s^i\right) < \infty,$$

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Applications to point processes

Outline of proof of Theorem 6:

1. Argue that it suffices to show that $\mathcal{E}((H \cdot M)^t - (H \cdot M)^u)$ is a martingale for $|t - u| \leq \varepsilon$.
2. Decompose μ into large and small parts and show a related decomposition for exponential martingales.
3. Apply two theorems of **(Lépingle & Mémin, 1978)** to obtain the result.

Applications to point processes

Example. Let $\mu_t^i \leq \alpha + \beta \sum_{j=1}^d N_{t-}^j$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

The Example shows that we may recover the classical affine criteria for non-explosion from the canonical case in the case of a general filtered space. This also extends the criterion from **(Gjessing et al. ,2010)** from an "$\mathcal{L}^p$"-criterion, $p > 1$, to an "$\mathcal{L}^p$"-criterion, $p \geq 1$.

Outline of proof. To use the first moment condition, use that $E \exp(\varepsilon X \log X)$ is finite when X is Poisson distributed and $0 < \varepsilon < 1$, choose $\varepsilon > 0$ such that $4\beta\varepsilon d < 1$. To use the second moment condition, use a Markov argument and that Poisson distributions have moments of all orders, choose $\varepsilon > 0$ such that $\beta\varepsilon d < 1$.

Applications to point processes

Example. Consider $A : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{R}^d$, $B : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d,d)$ and $\sigma : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d,d)$. Assume that $A(\eta,\cdot)$, $B(\eta,\cdot)$ and $\sigma(\eta,\cdot)$ are continuous and bounded for $\eta \in \mathbb{N}_0^d$. Assume that σ is positive definite. Assume that for $\eta \in \mathbb{N}_0^d$, there is $\delta, c > 0$ such that

$$\sup_{t\geq 0} \|A(\eta,t)\|_2 \leq c\|\eta\|_1^{1-\delta}$$

$$\sup_{t\geq 0} \|\sigma(\eta,t)\|_2 \leq c\|\eta\|_1^{(1-\delta)/2}$$

$$\sup_{t\geq 0} \|B(\eta,t)\|_2 \leq c.$$

Applications to point processes

Example, contined. Let X be a solution to

$$\mathrm{d}X_t = (A(N_t, Z_t) + B(N_t, Z_t)X_t)\,\mathrm{d}t + \sigma(N_t, Z_t)\,\mathrm{d}W_t,$$

where W is a d-dimensional $(\mathcal{F}_t)$ Brownian motion and $Z_t^i = t - T^i_{N^i_t}$, where T^i_n is the n'th event time of N^i. Let $\phi : \mathbb{R}^d \to \mathbb{R}^d_+$ be Lipschitz and put $\mu_t = \phi(X_t)$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

The example shows that we can use our results to construct counting processes where the intensity is driven by a SDE whose coefficients vary according to the history of the counting process.

Applications to point processes

Outline of proof. Note that conditionally on N, the intensity has the distribution of a Gaussian process. Apply bounds for $E\exp(c\|Z\|_2^{1+\varepsilon})$, with Z d-dimensionally Gaussian and $0 < \varepsilon < 1$, to obtain a bound for the conditional expectation

$$E\left(\exp\left(t\sum_{i=1}^{d}\mu_s^i\log_+\mu_s^i\right)\middle| N\right).$$

Use this to obtain a bound of the unconditional expectation varying continuously in s, $0 \le s \le t$. Apply Jensen's inequality and further estimates to obtain the result.

Applications to point processes

Example. Let $\phi_i : \mathbb{R} \to [0, \infty)$ for $i \le d$ and $h_{ij} : \mathbb{R}_+ \to \mathbb{R}$ for $i, j \le d$. Define

$$\mu_t^i = \phi_i \left(\sum_{j=1}^{d} \int_0^{t-} h_{ij}(t-s) \, \mathrm{d}N_s^j \right).$$

Assume that ϕ^i is Borel measurable, that $\phi_i(x) \le |x|$ and that h_{ij} is bounded. Then $\mathcal{E}(H \cdot M)$ is a martingale.

This is an example of a sufficient criterion for non-explosion for multidimensional Hawkes processes.

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Thank you!